TOWARD A THEORY OF PROPAGATION OF WAVES OF FINITE AMPLITUDE ALONG A RETARDING SYSTEM

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A great number of papers have been devoted to the propagation of a low-amplitude wave along a retarding system in the presence of an electron beam. An increase in the output power of such tubes (TWT, BWT, etc.) leads to the appearance of nonlinear effects. The propagation of electromagnetic waves of finite, but low amplitude has been examined in [1–5]. This makes it possible to take into account the influence of nonlinear effects by approximate methods. In this case, however, the limits of applicability and validity of these obtained solutions remain open. The possibility of obtaining particular but accurate solutions of the initial nonlinear system of equations takes on great interest in this connection. Brillouin [6], in particular, found an accurate solution in the form of a stationary wave, when all of the unknown functions (for example, the beam velocity) are dependent upon the space coordinate z and time t as the combination $\xi = z - Ut$, where U is the constant phase velocity.

Another, wider class of accurate solutions that describe the propagation of waves of finite amplitude along a retarding system with an electron beam is proposed below. It is assumed that the electron beam has only a longitudinal velocity component (strong focusing magnetic field), overtaking of one electron by another is absent, and that the effects of dissipation and the influence of the natural electromagnetic field of the beam on wave propagation are negligible. The initial system of equations of motion and continuity and the equations for the retarding line, under the above-mentioned assumptions, are reduced to

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = \eta\frac{\partial V}{\partial z}, \qquad \frac{\partial}{\partial t}\frac{I}{v} + \frac{\partial I}{\partial z} = 0 \qquad \left(\eta = \frac{|e|}{m}\right),$$

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \mp L\frac{\partial^2 I}{\partial z \partial t} = 0 \qquad \left(c = \frac{1}{\sqrt{LC}}\right), \quad (1)$$

where c and I are the velocity and current density of the electron beam; V the radio-frequency voltage in the retarding system; and L, C the self-inductance and capacitance per unit length.

In the last equation, the upper sign refers to TWT interaction, and the lower sign corresponds to the case of motion of the electron beam in the field of the backward wave of the retarding system.

A particular solution of system (1) will be a simple wave whose parameters I, v, V will be a function of only one of them [7], for example, v, i.e.,

$$I = I(v), \qquad V = V(v), \qquad v = v(z, t). \quad (2)$$

If we substitute (2) into (1), we reduce the particular solution to the form

$$v = \varphi\{z - U[I(v), v]t\},$$

$$\eta\frac{dV}{dv} = v - U, \qquad \frac{dI}{dv} = -\frac{IU}{v(v-U)}, \quad (3)$$

where φ is an arbitrary function, which is determined from the boundary or initial conditions, and U is the phase velocity of the wave, which satisfies the relation

$$v(v-U)^2(1 - U^2/c^2) = \pm\eta LIU^2. \quad (4)$$

From the latter relation, we see that the phase velocity of the wave U, unlike the assumption in [6], must be a function of the velocity and current density of the electron beam, i.e., $U = U(I, v)$. Relations (3) in this case are equivalent to the Riemann solution in gas dynamics, which describes the propagation of a sound wave of finite amplitude [8]. The dependence of the phase velocity U upon the velocity of the sound wave v results in the fact that the wave profile is distorted with propagation. Beginning with some moment in time, the solution becomes three-valued, which corresponds to the appearance of a shock wave. The influence of dissipation effects, which begin to play a considerable role in the presence of considerable distortion, can result in the fact that "reversing" of the front of the sound wave does not begin.

Similar distortion of the wave front must also be observed in the case in question. Unlike gas dynamics, dispersion and not dissipation plays the role of limiter of the steepness of the front, as in a rarefied plasma [9]. The simultaneous action of dispersion and the nonlinear effects leads in this case to the propagation of a stationary wave along the retarding system [9]. The form of this wave will differ considerably from the form of the tube input signal. In particular, if a monochromatic signal of frequency ω is delivered to the tube input, then harmonies $n\omega$ can be isolated from the output signal. It is to be expected that the amplitude of one of the harmonics, with proper selection of the parameters, may be equal to and even exceed the amplitude of the fundamental harmonic.

It must be noted, however, that for waves of fairly high amplitude, which exceeds the critical, a shock wave may appear even in the presence of dispersion, since the influence of the dispersion effects is insufficient to limit the growth of steepness.

In the general case, the propagation of a simple wave in implicit form is determined by system (3). Let us consider several particular cases, when the solution can be reduced to simple form. For definiteness, let us consider a BWT system.

A. Small nonlinear effects. If at the input of the tube at z = 0 a velocity perturbation $w = \cos\omega t$ is given whose amplitude is small in comparison with the constant velocity of the electron flux, then the expressions for v, I, and U are conveniently written as the sum of two terms

$$v = v_0 + w, \qquad I = i_0 - i, \qquad U = u_0 + u,$$

$$(w \ll v_0, \quad i \ll i_0, \quad u \ll u_0).$$

If we substitute this expansion into system (3) and satisfy the boundary condition at z = 0, we find that the expression for the perturbed velocity in the first approximation of the expansion in the small parameter, which characterizes the smallness of the nonlinear effects, is reduced to the form

$$w = a\cos\left\{\omega t - \frac{\omega z}{u_0}\left[1 + \alpha\cos\left(\omega t - \frac{\omega z}{u_0}\right)\right]\right\},$$

$$\alpha = \frac{3}{2}\frac{c^2 - u_0^2}{v_0 c^2 - u_0^3}. \quad (5)$$

Here u_0 is one of the roots of the equation

$$v_0(v_0 - u_0)^2(1 - u_0^2/c^2)^2 = \eta L i_0 u_0^2 \quad (6)$$

which coincides with the dispersion equation in the linear theory of backward-wave tubes. In the most important case, when the phase velocity of the wave is almost equal to the beam velocity ($u_0 \sim v_0$) but $c \neq v_0$, we obtain

$$u_0 \approx v_0\left[1 + \left(\frac{\eta L i_0 c^2}{v_0(c^2 - v_0^2)}\right)^{1/2}\right],$$

$$\alpha \approx \frac{3}{2}\frac{a}{v_0}\left[1 \pm \left(\frac{c - v_0^2}{c^2 - v_0^2}\right)^{1/2}\left(\frac{nLi_0}{v_0}\right)^{1/2}\right]. \quad (7)$$

From (5)–(7), we see that the solution is stable when the velocity of the electron beam is less than the phase velocity of the retarding system $v_0 < c$. The reverse inequality $v_0 > c$ must be satisfied for wave stability for TWT systems. If $u_0 \sim c = v_0$, then

$$u_0 \approx v_0(1 + \sqrt[3]{\eta L i_0 / 2v_0}), \qquad \alpha \approx \frac{a}{v_0}\left(1 - \sqrt[3]{\eta L i_0 / 2v_0}\right).$$

The above investigation shows that the influence of small nonlinear effects leads to distortion of the front and, in the final analysis, to the appearance of harmonics even in the first approximation of the expansion.

B. **Fast wave of finite amplitude** ($U \gg v$). In this limiting case the phase velocity of the wave

$$U \approx \pm c \sqrt{1 + \eta LI / v}. \tag{8}$$

If we substitute this relation into the last equation of system (3), we obtain

$$\sqrt{1 + \eta LI / v} - \ln \frac{1 + \sqrt{\eta LI / v + 1}}{1 - \sqrt{1 + \eta LI / v}} = \frac{v}{c} + A. \tag{9}$$

Here A is a constant, which is determined from the boundary or initial conditions. Relation (9) determines in implicit form the dependence of the current density upon the velocity of the electron beam. If we determine I(v) from (8), we can determine the dependence of the phase velocity of the wave upon $v = V(z, t)$.

In particular,

$$\frac{\eta LI}{v} \approx A + \frac{v}{c}, \quad U \approx \pm c \left(A + \frac{v}{c}\right)^{1/2} \text{ when } \frac{\eta LI}{v} \gg 1.$$

In the general case, system (3) cannot be solved in explicit form. Unlike the initial system, however, this system is simple and convenient for numerical integration by means of electronic computers.

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